

Symplectic singularities from the Poisson point of view

D. Kaledin*

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Introduction

In symplectic geometry, it is often useful to consider the so-called *Poisson bracket* on the algebra of functions on a C^∞ symplectic manifold M . The bracket determines, and is determined by, the symplectic form; however, many of the features of symplectic geometry are more conveniently described in terms of the Poisson bracket. When one turns to the study of symplectic manifolds in the holomorphic or algebro-geometric setting, one expects the Poisson bracket to be even more useful because of the following observation: the bracket is a purely algebraic structure, and it generalizes immediately to *singular* algebraic varieties and complex-analytic spaces.

The appropriate notion of singularities for symplectic algebraic varieties has been introduced recently by A. Beauville [B] and studied by Y.

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Namikawa [N1], [N2]. However, the theory of singular symplectic algebraic varieties is in its starting stages; in particular, to the best of our knowledge, the Poisson methods have not been used yet. This is the goal of the present paper.

Our results are twofold. Firstly, we prove a simple but useful structure theorem about symplectic varieties (Theorem 2.3) which says, roughly, that any symplectic variety admits a canonical stratification with a finite number of symplectic strata (in the Poisson language, a symplectic variety considered as a Poisson space has a finite number of symplectic leaves). In addition, we show that, locally near a stratum, the variety in question admits a nice decomposition into the product of the stratum itself and a transversal slice. Secondly, we study natural group actions on a symplectic variety and we prove that, again locally, a symplectic variety always admits a non-trivial action of the one-dimensional torus \mathbb{G}_m (Theorem 2.4). This is a rather strong restriction on the type of singularities a symplectic variety might have.

Unfortunately, the paper is much more eclectic than we would like. Moreover, one of the two main results is seriously flawed: we were not able to show that the \mathbb{G}_m -action provided by Theorem 2.4 has positive weights. However, all the results has been known to the author for a couple of years now, and it seems that any improvement would require substantially new methods. Thus we have decided to publish the statements “as is”.

In addition, we separately consider a special (and relatively rare) situation when a symplectic variety admits a crepant resolution of singularities. We prove that the geometry of such a resolution is very restricted: it is always semismall, and the Hodge structure on the cohomology of its fibers is pure and Hodge-Tate.

Our approach, for better or for worse, is to try to use Poisson algebraic methods as much as possible, getting rid of actual geometry at an early stage. The paper is organized as follows. In the first section we recall all the necessary definitions, both from the Poisson side of the story and from the theory of symplectic singularities. We also introduce two particular classes of Poisson schemes which we call *holonomic* and *locally exact*. In the second section we show that symplectic varieties give examples of Poisson schemes lying in both of these classes. The main technical tool here is the beautiful canonical resolution of singularities discovered in the last two decades (see, for example, [BM]). Then in Subsection 2.4, we study the geometry of crepant resolutions. The remainder of the paper is purely algebraic. First,

we prove the structure theorems for holonomic Poisson schemes. Then we show that a scheme is in addition locally exact, then it admits, again locally, a \mathbb{G}_m -action.

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1 Generalities on Poisson schemes.

Fix once and for all a base field k of characteristic $\text{char } k = 0$.

Definition 1.1. A *Poisson algebra* over the field k is a commutative algebra A over k equipped with an additional skew-linear operation $\{-, -\} : A \otimes A \rightarrow A$ such that

$$(1.1) \quad \begin{aligned} \{a, bc\} &= \{a, b\}c + \{a, c\}b, \\ 0 &= \{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\}, \end{aligned}$$

for all $a, b, c \in A$. An ideal $I \subset A$ is called a *Poisson ideal* if $\{i, a\} \in I$ for any $i \in I, a \in A$.

We will always assume that a Poisson algebra A has a unit element $1 \in A$ such that $\{1, a\} = 0$ for every $a \in A$.

Definition 1.2. A *Poisson scheme* over k is a scheme X over k equipped with a skew-linear bracket in the structure sheaf \mathcal{O}_X satisfying (1.1).

If A is a Poisson algebra over k , then $X = \text{Spec } A$ is a Poisson scheme. The reduction, every irreducible component, any completion and the normalization of a Poisson scheme are again Poisson schemes ([K2]). We will

say that a Poisson scheme is *local* if it is the spectrum of a local Poisson algebra A whose maximal ideal \mathfrak{m} is a Poisson ideal.

Let X be a Poisson scheme. For every local function f on X , the bracket $\{f, -\}$ is by definition a derivation of the algebra of functions, hence a vector field of X , denoted by H_f . Vector fields of the form H_f are called *Hamiltonian* vector fields. Moreover, the Poisson bracket is a derivation with respect to each of the two arguments. Therefore it can be expressed as

$$(1.2) \quad \{f, g\} = \Theta(df \wedge dg),$$

where

$$\Theta : \Lambda^2 \Omega_X^1 \rightarrow \mathcal{O}_X$$

is an \mathcal{O}_X -linear map. The map Θ is called the *Poisson bivector*. By the Jacobi identity part of (1.1), we have $H_f(\Theta) = 0$ for every Hamiltonian vector field H_f (Hamiltonian vector fields preserve the Poisson bivector). If X is smooth – for instance, if it is a point – then the cotangent sheaf Ω_X^1 is flat and Θ gives a skew-symmetric bilinear form on this sheaf.

Given a closed subscheme $Y \subset X$, we will say that Y is a *Poisson subscheme* if it is locally defined by a Poisson ideal in \mathcal{O}_X . Equivalently, a subscheme is Poisson if locally it is preserved by all Hamiltonian vector fields (in other words, all Hamiltonian vector fields are tangent to Y). In this case, Y inherits the structure of a Poisson scheme.

Definition 1.3. (i) An Noetherian integral Poisson scheme X over k with generic point $\eta \in X$ is called *generically non-degenerate* if the Poisson bivector Θ gives a non-degenerated skew-symmetric form on the cotangent module $\Omega^1(\eta/k)$.

(ii) A Noetherian integral Poisson scheme X is called *holonomic* if every integral Poisson subscheme $Y \subset X$ is generically non-degenerate.

In particular, a holonomic Poisson scheme X is itself generically non-degenerate. Moreover, every integral Poisson subscheme $Y \subset X$ of a holonomic Poisson scheme X is obviously holonomic. By definition, X itself and every such subscheme $Y \subset X$ must be even-dimensional over k . The normalization X' of a holonomic Poisson scheme is also holonomic (for every prime ideal $J' \subset A'$ in the normalization A' of a Poisson algebra A , the intersection $J = J' \cap A \subset A$ is a prime ideal in A , A'/J' is generically étale over A/J , and if J' is Poisson, J is obviously also Poisson).

The notion of a holonomic Poisson scheme has any meaning only for singular Poisson schemes; for a smooth scheme X it is vacuous because of the following.

Lemma 1.4. *Let X be a smooth Poisson scheme over k . Assume that X is holonomic. Then the Poisson bivector Θ is non-degenerate everywhere on X , and the only Poisson subscheme in X is X itself.*

Proof. Let $2n = \dim X$. The top degree power Θ^n of the Poisson bivector Θ is a section of the anticanonical bundle K_X^{-1} . Moreover, Θ is non-degenerate if and only if Θ^n is non-zero. Let $D \subset X$ be the zero locus of Θ^n . It is either empty, or a divisor in X . All the Hamiltonian vector fields preserve Θ , hence also Θ^n and D . Thus $D \subset X$ is a Poisson subscheme. But since X is by assumption holonomic, D must be even-dimensional – in particular, it cannot be a divisor. This proves the first claim. To prove the second, let $Y \subset X$ be a Poisson subscheme, and let $y \in Y$ be a closed point in the smooth part of Y . Then all Hamiltonian vector fields H_f are by definition tangent to Y at y . But since Θ is non-degenerate, Hamiltonian vector fields span the whole tangent space $T_y X$, and we have $Y = X$. \square

Holonomic Poisson schemes are the first special class of Poisson schemes that we will need in this paper. To introduce the second class, we give the following definition.

Definition 1.5. A Poisson algebra A over k is called *exact* if there exists a derivation $\xi : A \rightarrow A$ such that

$$(1.3) \quad \xi(\{a, b\}) = \{a, b\} + \{\xi(a), b\} + \{a, \xi(b)\}$$

for any $a, b \in A$.

Remark 1.6. This definition is motivated by the theory of *Poisson cohomology*, see e.g. [GK]. The Poisson bivector Θ defines a degree-2 Poisson cocycle on A , while any derivation $\xi : A \rightarrow A$ defines a degree-1 Poisson cochain. Equation (1.3) then says that $\Theta = \delta(\xi)$, where δ is the Poisson differential. We will not need this, so we do not give any details and refer the interested reader to [GK, Appendix].

Definition 1.7. A Poisson scheme X of finite type over k is called *locally exact* if for any closed point $x \in X$, the completed local ring $\hat{\mathcal{O}}_{X,x}$ is an exact Poisson k -algebra.

We will see in Section 3 that for holonomic Poisson schemes, local exactness passes to Poisson subschemes.

2 Symplectic singularities.

2.1 Statements. In this Section, assume that the base field k is a subfield $k \subset \mathbb{C}$ of the field of complex numbers. Let X be an algebraic variety – that is, an integral scheme of finite type over k . Assume that X is normal and even-dimensional, of dimension $\dim X = 2n$. Assume given a non-degenerate closed 2-form $\Omega \in \Omega^2(U)$ on the smooth open part $U \subset X$.

Definition 2.1 ([B],[N2]). One says that X is a *symplectic variety* – or, equivalently, that X has *symplectic singularities* – if the 2-form Ω extends to a (possibly degenerate) 2-form on a resolution of singularities $\tilde{X} \rightarrow X$.

Note that since for any two smooth birational varieties X_1, X_2 and any integer k we have

$$H^0(X_1, \Omega^k(X_1)) = H^0(X_2, \Omega^k(X_2)),$$

this definition does not depend on the choice of the resolution \tilde{X} (and indeed, one could have said “any resolution of singularities” right in the definition). Symplectic singularities are always canonical, hence rational (see [B]).

Any normal symplectic variety X is a Poisson scheme. Indeed, since X is normal, it suffices to define the bracket $\{f, g\}$ of any two local function $f, g \in \mathcal{O}_X$ outside of the singular locus. Thus we may assume that X is smooth. Then, since Ω is by assumption non-degenerate, it gives an identification $\mathcal{T}(X) \cong \Omega^1(X)$ between the tangent and the cotangent bundles on X , and this identification in turn gives a bivector $\Theta \in \Lambda^2 \mathcal{T}(X)$. It is well-known that Θ is a Poisson bivector for some Poisson structure if and only if Ω is closed. We will say that a smooth Poisson scheme X is *symplectic* if the Poisson structure on X is obtained by this construction from a non-degenerate closed 2-form Ω . A smooth Poisson scheme X is symplectic if and only if the Poisson bivector Θ is non-degenerate; the symplectic form Ω is uniquely defined by Θ . By Lemma 1.4, this is also equivalent to saying that the smooth Poisson scheme X is holonomic.

Exactness for symplectic varieties means exactly what one would expect.

Lemma 2.2. *Let $X = \operatorname{Spec} A$ be a normal affine symplectic variety. Then the Poisson algebra A is exact if and only if the symplectic form Ω satisfies*

$$\Omega = d\alpha$$

for some 1-form $\alpha \in \Omega^1(U)$ on the non-singular part $U \subset X$.

Proof. Using (1.2), it is easy to check that (1.3) in the symplectic case means exactly that

$$\xi(\Omega) = \Omega,$$

where ξ acts by Lie derivative. If A is exact, then by the Cartan homotopy formula we have

$$\Omega = \xi(\Omega) = \xi \lrcorner d\Omega + d(\xi \lrcorner \Omega) = d(\xi \lrcorner \Omega).$$

Conversely, if $\Omega = d\alpha$, then

$$\xi = \alpha \lrcorner \Theta \in \mathcal{T}(U)$$

satisfies (1.3). Since X is normal, ξ extends to a derivation of the whole algebra A . \square

We can now state the main two results of the paper.

Theorem 2.3. *Let X be a normal symplectic variety. Then there exists a finite stratification $X_i \subset X$ by irreducible Poisson subschemes such that all the open strata $X_i^\circ \subset X_i$ are smooth and symplectic. The only integral Poisson subschemes in X are closed strata. Moreover, for any closed point $x \in X_i^\circ \subset X$, the formal completion \widehat{X}_x of X at $x \in X$ admits a product decomposition*

$$(2.1) \quad \widehat{X}_x = \mathcal{Y}_x \times \widehat{X}_{i,x}^\circ,$$

where $\widehat{X}_{i,x}^\circ$ is the formal completion of the stratum X_i° at x and \mathcal{Y}_x is a local formal Poisson scheme and a symplectic variety. The product decomposition is compatible with the Poisson structures and the symplectic forms. \square

Theorem 2.4. *Let X be a normal symplectic variety. Then each of the transversal slices Y_x provided by Theorem 2.3 admits an action of the group \mathbb{G}_m such that for any point $\lambda \in \mathbb{G}_m(k) = k^*$*

$$\lambda \cdot \Omega_Y = \lambda^l \Omega_Y,$$

where Ω_Y is the symplectic form on the smooth open part of Y_x , and $l \neq 0$ is some integer. \square

The product decomposition (2.1) should be understood in the formal scheme sense (the spectrum of a *completed* tensor product). It is unfortunate that we have to pass to formal completions in the second part of Theorem 2.3 and in Theorem 2.4; however, this seems to be inevitable. We do not know to what extent the product decomposition (2.1) can be globalized. Note that the action in Theorem 2.4 must be non-trivial (so that its existence is a non-trivial restriction on the geometry of the transversal slice). The strata in Theorem 2.3 are what is known as *symplectic leaves* of the Poisson scheme X ; in particular, we prove that there is only a finite number of those. Theorem 2.3 is proved in Section 3, and Theorem 2.4 is proved in Section 4. Both are actually direct corollaries of the corresponding Poisson statements and the following theorem, which is the main result of this Section.

Theorem 2.5. *Let X be a normal symplectic variety. Then X is holonomic and locally exact. Moreover, the normalization of every Poisson subscheme $Y \subset X$ is also a symplectic variety.*

Before we turn to the proof of Theorem 2.5, we would like to note that the converse statement is false, at least in dimension 2 – there exist normal Poisson varieties which are holonomic and locally exact, but not symplectic. In fact, every weakly Gorenstein normal surface singularity is automatically Poisson and holonomic. It is often locally exact – for example, in the case when it admits a good \mathbb{C}^* -action. However, the only symplectic singularities in dimension 2 are rational double points.

On the other hand, if a variety X is non-singular outside of codimension 4, then every 2-form Ω on the smooth locus $U \subset X$ extends without poles to any smooth resolution $\tilde{X} \rightarrow X$ – this follows from the beautiful theorem of J. Steenbrink and D. van Straten [SvS], generalized by H. Flenner [F]. Thus every holonomic Poisson variety non-singular outside of codimension 4 is automatically a symplectic variety.

In practice, holonomic Poisson varieties usually *are* singular in $\text{codim } 2$, but these singularities are canonical – locally, we have a product of a smooth scheme and a transversal slice which is just a Du Val point. Thus in codimension 2 the singularity is symplectic. Unfortunately, the general extension theorem of Flenner-Steenbrink-van Straten says nothing at all about a situation of this type. There is one partial result, however. It has been proved by Y. Namikawa [N2, Theorem 4] that a normal variety X with *rational* singularities equipped with a symplectic form on the smooth part $X^{\text{reg}} \subset X$ is automatically a symplectic variety.

Remark 2.6. In [CF], F. Campana and H. Flenner defined the so-called *contact singularities* and proved that isolated contact singularities do not exist. They also conjectured that every contact singularity is the product of a symplectic singularity and an affine line. This conjecture should follow more or less directly from our Theorem 2.3 – one treats a contact singularity of dimension $2n+1$ as a symplectic singularity of dimension $2n+2$ equipped with a \mathbb{G}_m -action, and shows that our product decomposition is compatible with the \mathbb{G}_m -action. However, it seems that this statement is not as interesting as it might have been, because the notion of contact singularity is too restrictive. Essentially, Campana and Flenner require that a smooth resolution $\tilde{X} \rightarrow X$ admits a contact structure with trivial contact line bundle. In our opinion, especially from the point of view of our Theorem 2.4, it would be more interesting to allow contact line bundles which are not pulled back from line bundles on X .

2.2 Geometry of resolutions. Let X be a symplectic variety. Recall that X has rational singularities, so that for every smooth resolution $\pi : \tilde{X} \rightarrow X$ we have $R^i \pi_* \mathcal{O}_{\tilde{X}} = 0$ for $i \geq 1$. As usual, this implies in particular that $R^1 \pi_* \mathbb{Z} = 0$ in analytic topology (consider the exponential exact sequence on \tilde{X}). The form Ω on \tilde{X} defines a de Rham cohomology class $[\Omega] \in H_{DR}^2(\tilde{X})$. Recall that if we extend the field of definition from $k \subset \mathbb{C}$ to \mathbb{C} , then by the comparison theorem, $H_{DR}^2(\tilde{X})$ is isomorphic to the topological cohomology group $H^2(\tilde{X}, \mathbb{C})$, computed in analytic topology.

Lemma 2.7. *Let $\pi : \tilde{X} \rightarrow X$ be a smooth projective resolution of a symplectic variety X over \mathbb{C} , and let $[\Omega] \in H^2(\tilde{X}, \mathbb{C})$ be the cohomology class of the associated 2-form Ω on \tilde{X} . Then there exists a cohomology class $[\Omega_X] \in H^2(X, \mathbb{C})$ such that $[\Omega] = \pi^*[\Omega_X] \in H^2(\tilde{X}, \mathbb{C})$.*

Proof. The proof is an application of a beautiful idea of J. Wierzbka [Wi]. Consider the Leray spectral sequence of the map $\pi : \tilde{X} \rightarrow X$ and the associated three-step filtration on $H^2(\tilde{X}, \mathbb{C})$ whose graded pieces are subquotients of $H^p(X, R^q \pi_* \mathbb{C})$, $p+q=2$. We have to show that $[\Omega] \in \pi^*(H^2(X, \mathbb{C})) = \pi^*(H^2(X, \pi_* \mathbb{C})) \subset H^2(\tilde{X}, \mathbb{C})$. Since $R^1 \pi_* \mathbb{Z} = 0$, we also have $R^1 \pi_* \mathbb{C} = 0$; therefore the middle term in the associated graded quotient of $H^2(\tilde{X}, \mathbb{C})$ with respect to the Leray filtration vanishes, and it suffices to prove that the projection $H^2(\tilde{X}, \mathbb{C}) \rightarrow H^0(X, R^2 \pi_* \mathbb{C})$ annihilates $[\Omega]$. By proper base change, this is equivalent to proving that for every closed point $x \in X$, $[\Omega]$ restricts to 0 on the fiber $Z_x = \pi^{-1}(x) \subset \tilde{X}$ of the map $\pi : \tilde{X} \rightarrow X$.

Take such a fiber Z_x (with the reduced scheme structure). By [D], the cohomology groups $H^\bullet(Z_x, \mathbb{C})$ carry a natural mixed Hodge structure. To see it explicitly, one chooses a simplicial resolution Z_\bullet of the scheme Z_x by smooth proper schemes. Since $\pi : \tilde{X} \rightarrow X$ is projective, Z_x is also projective, and we can assume that Z_\bullet is a resolution by projective smooth schemes. Then the Hodge filtration F^\bullet on $H^\bullet(Z_x, \mathbb{C}) \cong H^\bullet(Z_\bullet, \mathbb{C}) = H_{DR}^\bullet(Z_\bullet)$ is induced by the stupid filtration on the de Rham complex $\Omega^\bullet(Z_\bullet)$.

Now, denote by F_X^\bullet the filtration induced on $H_{DR}^\bullet(\tilde{X})$ by the stupid filtration on the de Rham complex $\Omega^\bullet(\tilde{X})$. Since \tilde{X} is not compact, the corresponding spectral sequence does not degenerate. Nevertheless, the restriction map $P : H_{DR}^\bullet(\tilde{X}) \rightarrow H_{DR}^\bullet(Z_\bullet) \cong H^\bullet(Z_x, \mathbb{C})$, being induced by the natural map from Z_\bullet to the constant simplicial scheme \tilde{X} , obviously sends $F_X^l H^\bullet(\tilde{X})$ into $F^l H^\bullet(Z_x, \mathbb{C})$.

Since $R^{\geq 1}\pi_*\mathcal{O}_{\tilde{X}} = 0$, we may replace X with an affine neighborhood of the point $x \in X$ and assume that $H^{\geq 1}(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. Then $H_{DR}^l(\tilde{X}) = F_X^1 H_{DR}^l(\tilde{X})$ for any $l \geq 1$. Applying complex conjugation on $H_{DR}^l(\tilde{X}) \cong H^l(\tilde{X}, \mathbb{R}) \otimes \mathbb{C}$, we deduce that $H_{DR}^l(\tilde{X}) = \overline{F}_X^1 H_{DR}^l(\tilde{X})$, where \overline{F}_X^\bullet is the filtration complex-conjugate to F_X^\bullet . In particular, $[\Omega] \in \overline{F}_X^1 H_{DR}^2(\tilde{X})$. On the other hand, by definition we have $[\Omega] \in F_X^2 H_{DR}^2(\tilde{X})$.

Applying the restriction map $P : H^2(\tilde{X}, \mathbb{C}) \rightarrow H^2(Z_x, \mathbb{C})$, we deduce that $P([\Omega]) \in F^2 H^2(Z_x, \mathbb{C}) \cap \overline{F}^1(Z_x, \mathbb{C})$. But since the mixed Hodge structure on $H^2(Z_x, \mathbb{C})$ has weights ≤ 2 , this intersection is trivial. \square

Corollary 2.8. *Let $\pi : \tilde{X} \rightarrow X$ be a smooth projective resolution of a symplectic variety X . Then for every closed point $x \in X$, the restriction of the 2-form Ω on \tilde{X} to the formal neighborhood of the preimage $\pi^{-1}(x) \subset \tilde{X}$ is exact.*

Proof. We again take $k = \mathbb{C}$ and pass to the analytic topology. It suffices to find an open neighborhood $U \subset X$ of the point x such that Ω is exact on $\pi^{-1}(U) \subset \tilde{X}$. By Lemma 2.7, the cohomology class $[\Omega]$ of the form Ω comes from a cohomology class $[\Omega_X] \in H^2(X, \mathbb{C})$. Taking a small enough neighborhood $U \subset X$ of the point $x \in X$, we can insure that $[\Omega_X] = 0$ in $H^2(U, \mathbb{C})$. Thus we can take a neighborhood $U \subset X$ such that $[\Omega] = 0$ on $\tilde{U} = \pi^{-1}(U) \subset \tilde{X}$. Analyzing the Hodge-de Rham spectral sequence for \tilde{U} , we see that

$$\Omega = d_2\beta \mod d\left(H^0\left(\tilde{U}, \Omega_{\tilde{U}}^1\right)\right),$$

where d is the de Rham differential, d_2 is the second differential in the spectral sequence and β is a class in $H^1(\pi^{-1}(U), \mathcal{O}_{\tilde{U}})$. Since X has rational singularities, $\beta = 0$ and $\Omega = d\alpha$ for some 1-form α on \tilde{U} . \square

Lemma 2.9. *Let $\pi : \tilde{X} \rightarrow X$ be a smooth projective resolution of singularities of a symplectic variety X . Denote by $\Omega \in \Omega^2(\tilde{X})$ the symplectic form on the manifold \tilde{X} . Let $\sigma : Z \rightarrow U$ be a smooth map of smooth connected algebraic manifolds, and assume given a commutative square*

$$(2.2) \quad \begin{array}{ccc} Z & \xrightarrow{\eta} & \tilde{X} \\ \sigma \downarrow & & \downarrow \pi \\ U & \xrightarrow{\eta_0} & X. \end{array}$$

Then there exists a dense open subset $U_0 \subset U$ and 2-form $\Omega_U \in \Omega^2(U_0)$ on U_0 such that

$$(2.3) \quad \sigma^* \Omega_U = \eta^* \Omega$$

on $\sigma^{-1}(U_0) \subset Z$.

Proof. Consider the fibered product $\bar{Z} = U \times_X \tilde{X}$ equipped with the reduced scheme structure. Since $\tilde{X} \rightarrow X$ is projective, \bar{Z} is projective over U . Resolving the singularities, we can find a smooth, possibly non-connected scheme Z_0 projective over U and dominating every irreducible component of the scheme \bar{Z} . Consider the fibered product $Z_0 \times_{\bar{Z}} Z_0$. Resolving its singularities, we can find a smooth scheme Z_1 projective over U which dominates every irreducible component of $Z_0 \times_{\bar{Z}} Z_0$. Composing the map $Z_1 \rightarrow Z_0 \times_{\bar{Z}} Z_0$ with the natural projections, we obtain two maps $p_1, p_2 : Z_1 \rightarrow Z_0$. By the Bertini Theorem, we can replace U with a dense open subset so that the projection $Z_0 \rightarrow U$ and both maps $p_1, p_2 : Z_1 \rightarrow Z_0$ become smooth.

The schemes Z_1, Z_0 with the map $p_1, p_2 : Z_1 \rightarrow Z_0$ are the first terms of a simplicial hypercovering of the scheme \tilde{Z}/U , see [D]. In particular, let $Z_{1,u}, Z_{2,u}, \bar{Z}_u$ be the fibers of the schemes Z_1, Z_0, \bar{Z} over a closed point $u \in U$; then for every integer $l \geq 0$ we have an exact sequence

$$(2.4) \quad 0 \longrightarrow \mathrm{gr}_l^W H^l(\tilde{Z}_u, \mathbb{C}) \longrightarrow H^l(Z_{0,u}, \mathbb{C}) \xrightarrow{p_1^* - p_2^*} H^l(Z_{1,u}, \mathbb{C}),$$

where $\mathrm{gr}_p^W H^p(\tilde{Z}_u, \mathbb{C})$ is the weight- p part of the cohomology group $H^p(\tilde{Z}, \mathbb{C})$ with respect to the weight filtration associated to the mixed Hodge structure.

Now, the form Ω_U is uniquely defined by the conditions of the Lemma. Therefore it suffices to construct it locally on U , and even locally in étale topology. Moreover, since Z is smooth, it suffices to check (2.3) generically on U and on Z . Therefore, possibly replacing U with its étale cover and shrinking Z , we can assume that the map $\eta : Z \rightarrow \tilde{X}$ factors through $Z_0 \rightarrow \tilde{X}$. Thus it suffices to prove the Lemma when $Z \subset Z_0$ is a connected component of the scheme Z_0 .

To do this, we first note that by Lemma 2.7, the de Rham cohomology class $\eta^*[\Omega]$ of the form Ω vanishes on the fibers of the map $Z_0 \rightarrow U$. Since Z_0/U is projective, the Hodge to de Rham spectral sequence for the relative de Rham cohomology $H_{DR}^\bullet(Z_0/U)$ degenerates; therefore the form $\eta^*\Omega$ itself vanishes on fibers of the map $Z_0 \rightarrow U$. We conclude that for any closed point $u \in U$ and any tangent vector $\xi \in T_u U$, the relative 1-form $\alpha_\xi = \eta^*\Omega \lrcorner \xi \in H^0(Z_{0,u}, \Omega^1(Z_{0,u}))$ is well-defined (locally on Z_0 , lift ξ to a vector field on Z_0 , and check that α_ξ does not depend on the lifting). The same argument applies to the map $Z_1 \rightarrow U$. Moreover, since $\eta \circ p_1 = \eta \circ p_2 : Z_1 \rightarrow \tilde{X}$, we have

$$p_1^* \alpha_\xi = p_1^* \eta^* \Omega \lrcorner \xi = p_2^* \eta^* \Omega \lrcorner \xi = p_2^* \alpha_\xi.$$

Applying (2.4) with $l = 1$, we conclude that the cohomology class $[\alpha_\xi] \in H_{DR}^1(Z_{0,u})$ of the form α_ξ comes from a class in $H^1(\overline{Z}_u, \mathbb{C})$. But we know that $R^1\pi_*\mathbb{C} = 0$ on X , and by proper base change, this implies $H^1(\overline{Z}_u, \mathbb{C}) = 0$, so that $[\alpha_\xi] = 0$. Since the Hodge to de Rham spectral sequence for $Z_{0,u}$ degenerates, we conclude that $\alpha_\xi = 0$, for every closed point $u \in U$ and any tangent vector $\xi \in T_u U$. In other words, for every closed point $z \in Z \subset Z_0$ we have $\eta^*\Omega(\xi_1, \xi_2) = 0$ if at least one of the tangent vectors $\xi_1, \xi_2 \in T_z Z$ is vertical with respect to the map $\sigma : Z \rightarrow U$. This means that

$$\eta^*\Omega \in H^0(Z, \sigma^*\Omega^2(U)) \subset H^0(Z, \Omega^2(Z)).$$

To finish the proof, note that since Z is connected and projective over U , the natural map $\sigma^* : H^0(U, \Omega^2(U)) \rightarrow H^0(Z, \sigma^*\Omega^2(U))$ is one-to-one. \square

2.3 Proofs. To prove Theorem 2.5, we have to apply the results of the last Subsection to a particular resolution of singularities of the normal symplectic variety X . We will use the canonical resolution of singularities $\pi : \tilde{X} \rightarrow X$ constructed, for example, in [BM] or in [EH]. It enjoys the following two crucial properties:

- (i) The map $\pi : \tilde{X} \rightarrow X$ is one-to-one over the smooth part $X^o \subset X$.

(ii) Every vector field ξ on X lifts to a vector field $\tilde{\xi}$ on \tilde{X} .

Actually, the lifting property with respect to the vector fields is not claimed either in [BM] or in [EH] – the authors only prove equivariance with respect to automorphisms. However, the lifting property is easily deduced from this, for instance, along the lines of [K2, Lemma 2.2].

Proof of Theorem 2.5. Let X be a normal symplectic variety, and let $\pi : \tilde{X} \rightarrow X$ be the canonical resolution of singularities of the variety X . Since all the claims of the Theorem are local, we may assume that X is affine.

To prove that X is locally exact, it suffices to apply Corollary 2.8 to the resolution \tilde{X} and then invoke Lemma 2.2.

To prove that X is holonomic, we use an idea that essentially goes back to [BHL]. For any locally closed subscheme $U \subset X$, denote by $\pi^{-1}(U) \subset \tilde{X}$ its preimage under the map $\pi : \tilde{X} \rightarrow X$ equipped with the reduced scheme structure. Let $Y \subset X$ be an integral Poisson subscheme, and let $U \subset Y$ be the open dense subset such that U is smooth and the map $\pi : \pi^{-1}(U) \rightarrow U$ is generically smooth on $\pi^{-1}(u) \subset \pi^{-1}(U)$ for any closed point $u \in U$. It suffices to prove that the Poisson bivector Θ is non-degenerate on the cotangent space $T_y^*(Y)$ for every point $y \in U$. Fix such a point y . If Θ is degenerate, then some non-trivial covector in $T_y^*(Y)$ lies in its annihilator. In other words, there exists a function f on X such that $df \neq 0$ at $y \in U$, but the Hamiltonian vector field

$$H_f = df \lrcorner \Theta$$

vanishes at the point $y \in Y \subset X$. On the other hand, generically on X we have a well-defined symplectic form Ω , and we have

$$df = H_f \lrcorner \Omega.$$

Since the resolution \tilde{X} is canonical, the vector field H_f lifts to a vector field \tilde{H}_f on \tilde{X} . Generically we have

$$\pi^*(df) = \tilde{H}_f \lrcorner \Omega.$$

But \tilde{X} is smooth and Ω is defined everywhere. Therefore this equality also holds everywhere – in particular, on every connected component of the smooth part of $\pi^{-1}(U)$. By Lemma 2.9, we can replace U with a dense open subset so that on such a connected component we have $\Omega = \pi^*\Omega_0$ for some 2-form Ω_0 on U . Since H_f preserves $U \subset X$, the vector field \tilde{H}_f preserves (that is, is tangent to) its preimage $\pi^{-1}(U)$, and we have

$$\tilde{H}_f \lrcorner \Omega = H_f \lrcorner \Omega_0.$$

Since H_f vanishes at y , we conclude that $df = 0$ in every point in the smooth part of $\pi^{-1}(y) \subset X$. By assumption $df \neq 0$ at $y \in U$, and this contradicts our smoothness assumptions on the map $\pi : \pi^{-1}(U) \rightarrow U$.

Finally, we have to prove that the normalization Y^{nrm} of an irreducible Poisson subscheme Y is a symplectic variety. We note that since we already know that X is holonomic, Y is generically symplectic. Moreover, every Poisson subscheme in Y is also a Poisson subscheme in X . Therefore Y itself is holonomic, and so is its normalization Y^{nrm} . By Lemma 1.4 this means that the regular part $Y^{reg} \subset Y^{nrm}$ carries a symplectic form. We have to prove that this form extends to a smooth resolution of the variety Y^{nrm} .

To do this, note that generically on Y and locally in étale topology, the projection $\pi^{-1}(Y) \rightarrow Y$ admits a section. More precisely, taking a sufficiently small smooth open dense subset $U \subset Y$, we can assume that there exists a Galois cover $\kappa : U' \rightarrow U$ and a map $\sigma : U' \rightarrow \tilde{X}$ such that $\sigma \circ \pi = \kappa : U' \rightarrow U \subset X$. By Lemma 2.9, shrinking U even further we can assume that $\sigma^*\Omega = \kappa^*\Omega_U$ for some 2-form Ω_U on U . We will prove that (1) this form Ω_U extends to a resolution \tilde{Y} of the variety Y , and (2) at least generically on Y , the form Ω_U coincides with the form given by the Poisson structure on Y .

Step 1: Ω_U extends to a form on a resolution $\tilde{Y} \rightarrow Y$.

Let Y' be the normalization of the scheme Y in the Galois cover $U' \rightarrow U$, and consider the fibered product $\tilde{X} \times_Y Y'$ equipped with the reduced scheme structure. Let $\overline{\sigma(U')} \subset \tilde{X} \times_Y Y'$ be the closure of the image of the section $\sigma : U' \rightarrow \tilde{X} \times_Y Y'$, and let \tilde{Y}' be a smooth projective $Gal(U'/U)$ -equivariant resolution of singularities of the closure $\overline{\sigma(U')}$.

We have a $Gal(U'/U)$ -equivariant smooth resolution $\tilde{Y}' \rightarrow Y'$ and a map $\sigma : \tilde{Y}' \rightarrow \tilde{X}$. By assumption, over $U' \subset Y'$ the 2-form $\sigma^*\Omega$ on \tilde{Y}' coincides with $\kappa^*\Omega_U$. In particular, the 2-form $\sigma^*\Omega$ is $Gal(U'/U)$ -invariant. The quotient $\tilde{Y}_0 = \tilde{Y}'/Gal(U'/U)$ is a normal algebraic variety equipped with a projective birational map onto Y .

In general, let $f : Z' \rightarrow Z$ be a finite morphism between normal algebraic varieties such that Z' is smooth and equipped with an action of a finite group G , and $f : Z' \rightarrow Z$ is generically a Galois cover with Galois group G . Then for any $p \geq 0$, any G -invariant p -form α on Z' gives by descent a p -form on an open smooth subset of the variety Z , and this form extends to any smooth projective resolution $\tilde{Z} \rightarrow Z$ (this is well-known; for a sketch of a proof see e.g. [K1, Lemma 3.3]). In particular, take $p = 2$, $Z' = \tilde{Y}'$ and $G = Gal(U'/U)$. Then the $Gal(U'/U)$ -invariant 2-form $\sigma^*\Omega$ on \tilde{Y}' gives a

2-form Ω on any smooth projective resolution $\tilde{Y} \rightarrow \tilde{Y}_0$.

Since generically on \tilde{Y}' we have $\sigma^*\Omega = \kappa^*\Omega_U$, we conclude that the 2-form Ω_U on $U \subset Y$ extends to a smooth projective resolution $\tilde{Y} \rightarrow \tilde{Y}_0 \rightarrow Y$.

Step 2: Ω_U is compatible with the Poisson structure on Y .

Since the Poisson structure on U is non-degenerate, the tangent bundle $\mathcal{T}(U)$ is generated by Hamiltonian vector fields. Since the map $\kappa : U' \rightarrow U$ is étale, all vector fields on U lift to vector fields on U' . Thus it suffices to check that for every two Hamiltonian vector fields

$$H_f = df \lrcorner \Theta, \quad H_g = dg \lrcorner \Theta,$$

we have $(\sigma^*\Omega)(H_f, H_g) = \{f, g\}$ on U' . Again, both H_f and H_g lift to vector fields \tilde{H}_f, \tilde{H}_g on \tilde{X} , and by Lemma 2.9 we have

$$(\sigma^*\Omega)(H_f, H_g) = \sigma^*\left(\Omega\left(\tilde{H}_f, \tilde{H}_g\right)\right).$$

Thus it suffices to check that

$$\Omega\left(\tilde{H}_f, \tilde{H}_g\right) = \pi^*\{f, g\}.$$

But this equation makes sense everywhere on \tilde{X} . Since \tilde{X} is reduced, it suffices to check it generically, where it follows from the definition of the form Ω . \square

2.4 Symplectic resolutions. Assume now given a symplectic variety X and a smooth projective resolution $\tilde{X} \rightarrow X$ which is *crepant* – in this context, it means that the canonical bundle $K_{\tilde{X}}$ is trivial. Since the top degree $\Omega^{\frac{1}{2}\dim X}$ of the symplectic form is a section of $K_{\tilde{X}}$, the form $\Omega_{\tilde{X}}$ in this case must be non-degenerate everywhere, not only at the generic point of the scheme \tilde{X} . It turns out that this imposes strong restrictions on the geometry of \tilde{X} . We start with the following general fact.

Lemma 2.10. *Let X, Y be algebraic varieties over k , assume that X is smooth, and let $\tau : X \rightarrow Y$ be a projective map. Then $R^p\tau_*\Omega_X^q = 0$ whenever $p + q > \dim X$.*

Proof. (I am grateful to H. Esnault and E. Viehweg for suggesting and explaining this proof to me.)

The claim is local in Y , so that we may assume that Y is affine. Choose a projective variety \bar{Y} , a smooth projective variety \bar{X} and a projective map

$\tau : \overline{X} \rightarrow \overline{Y}$ so that $Y \subset \overline{Y}$ is a dense open subset in \overline{Y} , $X = \pi^{-1}(Y) \subset \overline{X}$ is a dense open subset in \overline{X} , $\overline{\tau} : \overline{X} \rightarrow \overline{Y}$ extends the given map $\tau : X \rightarrow Y$, and the complement $E = \overline{X} \setminus X$ is a simple normal crossing divisor in \overline{X} . Choose an ample line bundle M on \overline{Y} . Consider the logarithmic de Rham complex $\Omega_{\overline{X}}^{\bullet}(\log E)$. Let $l \gg 0$ be an integer large enough so that the sheaves

$$(2.5) \quad R^p \overline{\tau}_* \Omega_{\overline{X}}^q(\log E)(-E) \otimes M^{\otimes l}$$

are acyclic and globally generated for all p, q . Replace M with $M^{\otimes l}$. Then by [EV, Corollary 6.7], we have

$$H^p \left(\overline{X}, \Omega_{\overline{X}}^q(\log E) \otimes \tau^* M^{-1} \right) = 0$$

whenever $p + q < \dim X - r(\tau)$. Here $r(\tau)$ is a certain constant defined in [EV, Definition 4.10]; although the definition is textually different, it is immediately obvious that $\dim X + r(\tau) = \dim X \times_Y X$. By Serre duality (see [EV, Remark 6.8 b)),

$$H^p \left(\overline{X}, \Omega_{\overline{X}}^q(\log E)(-E) \otimes \tau^* M \right) = 0$$

whenever $p + q > \dim X + r(\tau) = \dim X \times_Y X$. Since the sheaves (2.5) are acyclic, the Leray spectral sequence for these cohomology groups collapses, and we conclude that

$$H^0 \left(R^p \overline{\tau}_* \Omega_{\overline{X}}^q(\log E)(-E) \otimes M \right) = 0$$

wherever $p + q > \dim X \times_Y X$. Since the sheaves (2.5) are globally generated, this implies $R^p \overline{\tau}_* \Omega_{\overline{X}}^q(\log E)(-E) = 0$. Restricting to $Y \subset \overline{Y}$, we get the claim. \square

This can be applied to the symplectic situation because of the following.

Lemma 2.11. *Let $\pi : \tilde{X} \rightarrow X$ be a projective birational map from a smooth variety \tilde{X} with $K_{\tilde{X}} = 0$ to a symplectic variety X . Then the map $\pi : \tilde{X} \rightarrow X$ is semismall, in other words, $\dim \tilde{X} \times_X \tilde{X} = \dim X$.*

Proof. For any $p \geq 0$, let $X_p \subset X$ be the closed subvariety of points $x \in X$ such that $\dim \pi^{-1}(x) \geq p$. It suffices to prove that $\text{codim } X_p \geq 2p$. By Lemma 2.9, there exists an open dense subset $U \subset X_p$ such that the restriction Ω_F of the form $\Omega = \Omega_{\tilde{X}}$ onto every connected component F of the smooth part of the set-theoretic preimage $\pi^{-1}(U) \subset \tilde{X}$ satisfies

$$\Omega_F = \pi^* \Omega_U$$

for some 2-form $\Omega_U \in H^0(U, \Omega_U^2)$. Therefore the rank $\text{rk } \Omega_F$ satisfies $\text{rk } \Omega_F \leq \dim U$. On the other hand, since $K_{\tilde{X}}$ is a trivial line bundle, the form Ω is non-degenerate on \tilde{X} , and we have $\text{rk } \Omega_F \geq \dim F - \text{codim } F$. By definition of $X_p \subset X$, we can choose a component F such that $\dim F = \dim U + p$. Then together these two inequalities give $\text{codim } X_p = \text{codim } F + p \geq \dim F - \dim U + p = 2p$, as required. \square

Theorem 2.12. *Let $\pi : \tilde{X} \rightarrow X$ be a projective birational map with smooth and symplectic \tilde{X} . Let $x \in X$ be a closed point, and let $E_x = \pi^{-1}(x) \subset \tilde{X}$ be the set-theoretic fiber over the point x . Then for odd k we have $H^k(E_x, \mathbb{C}) = 0$, while for even $k = 2p$ the Hodge structure on $H^k(E_x, \mathbb{C})$ is pure of weight k and Hodge type (p, p) .*

Lemma 2.13. *Let p be an integer, and let V be an \mathbb{R} -mixed Hodge structure with Hodge filtration F^\bullet and weight filtration W_\bullet . Assume that $W_{2p}V = V$ and $F^pV = V$. Then V is a pure Hodge-Tate structure of weight $2p$ (in other words, every vector $v \in V$ is of Hodge type (p, p)).*

Proof. Since $V = F^pV$, the same is true for all associated graded pieces of the weight filtration on V . Therefore we may assume that V is pure of weight $k \leq 2p$. If $k < 2p$, we must have $V = F^pV \cap \overline{F^pV} = 0$, which implies $V = 0$. If $k = 2p$, the same equality gives $V = V^{p,p}$. \square

Proof of Theorem 2.12. By Lemma 2.11, Lemma 2.10 applies to $\pi : \tilde{X} \rightarrow X$ and shows that

$$(2.6) \quad R^p\pi_*\Omega_{\tilde{X}}^q = 0$$

whenever $p + q > \dim \tilde{X}$. Since \tilde{X} is symplectic, we have an isomorphism $\mathcal{T}_{\tilde{X}} \cong \Omega_{\tilde{X}}^1$ between the tangent and the cotangent bundle on \tilde{X} . This implies that $\Omega_{\tilde{X}}^q \cong \Omega_{\tilde{X}}^{\dim \tilde{X} - q}$, and (2.6) also holds whenever $p > q$.

Denote by \mathfrak{X} the completion of the variety \tilde{X} in the closed subscheme $E_x = \pi^{-1}(x) \subset \tilde{X}$. Since the map $\pi : \tilde{X} \rightarrow X$ is proper, by proper base change the group

$$H^p(\mathfrak{X}, \Omega_{\mathfrak{X}}^q)$$

for any p, q coincides with the completion of the stalk of the sheaf $R^p\pi_*\Omega_{\tilde{X}}^q$ at the point $x \in X$. Therefore $H^p(\mathfrak{X}, \Omega_{\mathfrak{X}}^q) = 0$ whenever $p > q$. The stupid filtration on the de Rham complex $\Omega_{\mathfrak{X}}^\bullet$ of the formal scheme \mathfrak{X} induces a decreasing filtration F^\bullet on the de Rham cohomology groups $H_{DR}(\mathfrak{X})$

which we call the weak Hodge filtration. Of course, the associated spectral sequence does not degenerate. Nevertheless, since $H^p(\mathfrak{X}, \Omega_{\mathfrak{X}}^q) = 0$ when $p > q$, we have $H_{DR}^k(\mathfrak{X}) = F^p H_{DR}^k(\mathfrak{X})$ whenever $k \leq 2p$.

It is well-known that the canonical restriction map

$$H_{DR}^{\bullet}(\mathfrak{X}) \rightarrow H^{\bullet}(E_x, \mathbb{C})$$

is an isomorphism. By definition (see [D]), to obtain the Hodge filtration on the cohomology groups $H^{\bullet}(E_x)$, one has to choose a smooth simplicial resolution E_x^{\bullet} for the variety E_x and take the usual Hodge filtration on $H^{\bullet} DR(E_x^{\bullet})$. The embedding $E_x \rightarrow \mathfrak{X}$ gives a map $E_x^{\bullet} \rightarrow \mathfrak{X}^{\bullet}$, where \mathfrak{X}^{\bullet} is \mathfrak{X} considered as a constant simplicial variety. The corresponding restriction map

$$H_{DR}^{\bullet}(\mathfrak{X}) \rightarrow H_{DR}^{\bullet}(E_x^{\bullet})$$

is also an isomorphism, and it sends the weak Hodge filtration on the left-hand side into the usual Hodge filtration on the right-hand side. We conclude that $H^k(E_x) = F^p(E_x)$ whenever $k \leq 2p$. It remains to recall that by definition, we have $H^k(E_x) = W_k H^k(E_x)$, and apply Lemma 2.13. \square

To conclude this subsection, we would like to note that in the particular case when $X = T^*(G/B)$ is the Springer resolution of the nilpotent cone $Y = \mathcal{N} \subset \mathfrak{g}^*$ in the coadjoint representation \mathfrak{g}^* of a semisimple algebraic group G , Theorem 2.12 has been already proved by C. de Concini, G. Lusztig and C. Procesi in [dCLP]. They proceed by a direct geometric argument. As a result, they obtain more: not only do the cohomology groups carry a Hodge structure of Hodge-Tate type, but in fact they are spanned by cohomology classes of algebraic cycles. This is true even for cohomology groups with integer coefficients. Motivated by this, we propose the following.

Conjecture 2.14. *In the assumptions of Theorem 2.12, the cohomology groups $H^k(E_x, \mathbb{Z})$ are trivial for odd k , and are spanned by cohomology classes of algebraic cycles for even k .*

We also expect that an analogous statement holds for l -adic cohomology groups, possibly even over fields of positive characteristic.

3 Stratification and product decomposition.

We now turn to the algebraic study of Poisson schemes.

Proposition 3.1. *Let X be a Noetherian integral Poisson scheme over k . Assume that X is excellent as a scheme and holonomic as a Poisson scheme. Then there exists a stratification $X_i \subset X$ by Poisson subschemes such that the open parts X_i° of the strata are smooth and symplectic. The stratification is canonical – in particular, it is preserved by all automorphisms of the scheme X and by all vector fields. The only integral Poisson subschemes in X are the irreducible components of the closed strata X_i .*

Proof. Let $Y \subset X$ be singular locus of the scheme X . Since X is excellent, Y is a proper closed subscheme preserved by all automorphisms of X and by all vector fields. In particular, it is preserved by all Hamiltonian vector fields. This means that $Y \subset X$ is a Poisson subscheme. It is automatically holonomic. Since $\dim Y < \dim X$, we can stratify it by induction. Thus it suffices to stratify the smooth part $X \setminus Y \subset X$. In this case all the claims follow from Lemma 1.4. \square

Note that this immediately implies that if a holonomic Poisson scheme X is locally exact, then every Poisson subscheme $Y \subset X$ is also locally exact. Indeed, every such subscheme must be a closed stratum X_i , hence it is preserved by *all* locally defined vector fields on X – in particular, by a vector field ξ satisfying (1.3).

Next, we construct the product decomposition (2.1). We need the following general result.

Lemma 3.2. *Let M be a vector space equipped with a decreasing filtration $F^p M$, $p \geq 0$ such that $\text{codim } F^p M < \infty$ and M is complete with respect to the topology induced by F^\bullet . Assume that M is a module over the algebra $A = k[[x_1, \dots, x_n]]$ of formal power series in n variables, and that $x_i \cdot F^p M \subset F^{p+1} M$ for every i, p . Finally, assume that the module M is equipped with a flat connection $\nabla : M \rightarrow M \otimes \Omega_A^1$, and let $M^\nabla \subset M$ be the subspace of flat sections. Then the natural map*

$$M^\nabla \hat{\otimes} A \rightarrow M$$

from the completed tensor product $M \hat{\otimes} A$ to M is an isomorphism.

Proof. This is completely standard, but under an additional assumption that M is finitely generated. We give a proof to show that our assumptions are in fact sufficient.

Consider the algebra D of differential operators $A \rightarrow A$; as a vector space, $D \cong A[\xi_1, \dots, \xi_n]$, where ξ_i denotes the differential operator $\frac{\partial}{\partial x_i}$. An

A -module equipped with a flat connection ∇ is the same as a left D -module (the generator ξ acts by covariant derivative with respect to $\frac{\partial}{\partial x_i}$). Since M is a cocompact topological vector space, we have $M = N^*$, where N is the (discrete) vector space of continuous linear maps $M \rightarrow k$. The filtration F induces an increasing filtration $F_p N$, $p \geq 0$ such that $\dim F_p N < \infty$. The maps $x_\bullet, \xi_\bullet : M \rightarrow M$ induce by duality maps $x_\bullet, \xi_\bullet : N \rightarrow N$ satisfying the same commutation relations. We have $x_i \cdot F_{p+1} N \subset F_p N$ for every i, p ; therefore every element $a \in N$ is annihilated by a high power of every x_i , and N becomes a left D -module.

Let $N^o \subset N$ be the common kernel of multiplication by x_1, \dots, x_n . We first prove that for any filtered left D -module N satisfying $x_i \cdot F_{p+1} N \subset F_p N$, the natural map

$$a_{N, N^o} : N^o \otimes k[\xi_1, \dots, \xi_n] \rightarrow N$$

induced by the $k[\xi_1, \dots, \xi_n]$ -module structure on N is an isomorphism.

By induction, it suffices to consider the case $n = 1$. Indeed, let $N' \subset N$ be the kernel of multiplication by x_1 . Then N' carries a natural structure of a filtered module over $k[[x_2, \dots, x_n]]$ equipped with a flat connection, and it satisfies all our assumptions. The map a_{N, N^o} factors as

$$N^o \otimes k[\xi_1, \dots, \xi_n] \xrightarrow{a_{N', N^o} \otimes \text{id}} N' \otimes k[\xi_1] \xrightarrow{a_{N, N'}} N,$$

we know by the inductive assumption that a_{N', N^o} is an isomorphism, and we have to prove that $a_{N, N'}$ is also an isomorphism. Thus we may forget about x_i, ξ_i for $i \geq 2$ and assume $n = 1$, $N^o = \text{Ker } x_1 \subset N$.

To simplify notation, let $a_N = a_{N, N^o}$. Note that for every $k[[x_1]]$ -module N satisfying our assumptions, the kernel $\text{Ker } x_1$ must be non-trivial (for instance, it contains the smallest non-trivial term in the filtration $F_\bullet N$). In particular, this applies to the kernel $\text{Ker } a_N \subset N^o \otimes k[\xi_1]$ of the map a_N . But the kernel of x_1 on $\text{Ker } a_N \subset N^o \otimes k[\xi_1]$ coincides with $\text{Ker } a_N \cap \text{Ker } x_1 \subset N^o \otimes k[\xi_1]$, which in turn is equal to the kernel of the map a_N on $N^o = \text{Ker } x_1 \subset N \otimes k[\xi_1]$; since by definition this kernel is trivial, we must also have $\text{Ker } a_N = 0$. We conclude that a_N is injective. It remains to check that N is generated by $\text{Ker } x_1$ as a $k[\xi_1]$ -module. Denote $N_p = \text{Ker } x_1^p \subset N$. By assumption $F_p N \subset N_p$, so that $N = \bigcup N_p$. Thus by induction it suffices to prove that

$$N_p \subset N_{p-1} + k[\xi_1] \cdot N_1.$$

This is immediate. For every element $m \in N_p$, let

$$m_0 = m - \frac{1}{(p-1)!} \xi_1^{p-1} x_1^{p-1} m.$$

Since $x_1\xi_1 - \xi_1x_1 = \text{id}$, we have $x_1^{p-1}m_0 = 0$, and $m_0 \in N_{p-1}$.

Now, we have proved that $N \cong V \otimes k[\xi_1, \dots, \xi_n]$ for some vector space $V = N^o$; since x_i vanishes on N^o , both the operators x_\bullet and the operators ξ_\bullet act on $V \otimes k[\xi_1, \dots, \xi_n]$ via the second factor. Therefore

$$M = N^* \cong V^* \widehat{\otimes} A,$$

where again x_\bullet and ξ_\bullet act on the product via the second factor. To prove the Lemma, it suffices to show that the natural map $M^\nabla \rightarrow M$ identifies M^∇ with $V^* \otimes 1 \subset V^* \widehat{\otimes} A$. If V^* is one-dimensional, this is obvious: A^∇ is indeed the line spanned by $1 \in A$. But the completed tensor product functor $W \mapsto W \widehat{\otimes} A$ is exact and commutes with arbitrary inverse limits, and in particular, with arbitrary products; the flat sections functor $M \mapsto M^\nabla$ is left-exact and therefore also commutes with arbitrary products. Since every cocompact vector space V^* is a (possibly infinite) product of one-dimensional vector spaces, we are done. \square

Proposition 3.3. *Let A be a complete Poisson local algebra over k with maximal ideal $\mathfrak{m} \subset A$, and assume given a prime Poisson ideal $J \subset A$ such that the quotient A/J is a regular complete local algebra with non-degenerate Poisson structure. Then there exists a complete local Poisson algebra B and a Poisson isomorphism*

$$(3.1) \quad A \cong B \widehat{\otimes}_k (A/J)$$

between the algebra A and the completed tensor product of the algebras A/J and B . Moreover, the Poisson scheme $\text{Spec } A$ is holonomic if and only if the Poisson scheme $\text{Spec } B$ is holonomic, and the Poisson algebra A is exact if and only if the Poisson algebra B is exact.

Proof. For every integer $d \geq 1$, denote by W_d the power series algebra $k[[x_1, \dots, x_d, y_1, \dots, y_d]]$ with the standard Poisson structure induced by the symplectic form $dx_1 \wedge dy_1 + \dots + dx_d \wedge dy_d$. We first prove that there exists a Poisson isomorphism

$$(3.2) \quad A \cong W_1 \widehat{\otimes} A'$$

for some complete local Poisson algebra A' . By the formal Darboux Theorem, there exists a Poisson isomorphism $A/J \cong W_d$, where $2d = \dim A/J$. Fix arbitrary liftings $f, g \in A$ of $x_1, y_1 \in A/J$. Then $\{f, g\} = 1 \pmod{J}$. We claim that there exist a series of functions $f_l \in J^l$, $l \geq 1$ such that

$$(3.3) \quad \{f, g + g_1 + \dots + g_l\} = 0 \pmod{J^{l+1}}.$$

Indeed, let $\xi(a) = \{a, g\}$ for $a \in A$; then by induction on l it suffices to prove that $\xi : J^l/J^{l+1} \rightarrow J^l/J^{l+1}$ is surjective. However, on J^l/J^{l+1} we have $\xi(fa) = f\xi(a) + a$, so that ξ induces a flat connection on J^l/J^{l+1} considered as a $k[[f]]$ -module. Therefore we can equip J^l/J^{l+1} with \mathfrak{m} -adic filtration and apply Lemma 3.2.

Having chosen a sequence g_l as in (3.3), replace g with

$$g' = g + g_1 + \dots,$$

so that $\{f, g'\} = 1$ in the algebra A , and let $\xi_1(a) = \{f, a\}$, $\xi_2(a) = \{g', a\}$ for any $a \in A$. Then we have $k[[f, g']] \cong W_1$, and ξ_1 and ξ_2 induce a flat connection on A considered as a $k[[f, g']]$ -module. Let $A' = A_0 = \text{Ker } \xi_1 \cap \text{Ker } \xi_2$ and apply Lemma 3.2 to A equipped with \mathfrak{m} -adic filtration. The space A' is obviously a complete Poisson algebra, and by Lemma 3.2 we indeed have the isomorphism (3.2). To finish the proof of the first claim, apply induction on $2d = \dim A/J$.

To prove the second claim, note that by Lemma 3.2 every Poisson ideal $I \subset A$ is equal to $I_0 \otimes A/J$ for some Poisson ideal $I_0 \subset B$. Thus every Poisson subscheme in $Y \subset \text{Spec } A$ is the (completed) product $Y_0 \times \text{Spec } A/J$ of a Poisson subscheme $Y_0 \subset \text{Spec } B$ with the non-degenerate smooth Poisson scheme $\text{Spec } A/J$. This implies that $\text{Spec } A$ is holonomic if and only if $\text{Spec } B$ is holonomic.

Finally, the algebra $A/J \cong W_d$ is obviously exact – for instance, the Euler vector field

$$\xi_e = \frac{1}{2} \frac{\partial}{\partial x_1} + \dots + \frac{1}{2} \frac{\partial}{\partial x_d} + \frac{1}{2} \frac{\partial}{\partial y_1} + \dots + \frac{1}{2} \frac{\partial}{\partial y_d}$$

satisfies (1.3). Thus if B is exact, with a derivation $\xi_0 : B \rightarrow B$ satisfying (1.3), then the derivation

$$\xi = \xi_0 \otimes \text{id} + \text{id} \otimes \xi_e$$

of the algebra $A \cong B \otimes (A/J)$ also satisfies (1.3), and A is exact. Conversely, assume that we are given a derivation $\xi : A \rightarrow A$ satisfying (1.3). The product decomposition (3.1) gives in particular a canonical embedding $A/J \subset A$, thus a direct some decomposition $A \cong J \oplus (A/J)$. The restriction $\bar{\xi} : A/J \rightarrow A$ of the map ξ to $A/J \subset A$ decomposes as

$$\bar{\xi} = \xi' + \xi_1,$$

where $\xi_1 : A/J \rightarrow A/J$ is a derivation satisfying (1.3), and $\xi' : A/J \rightarrow J$ is a derivation which is also a derivation with respect to the Poisson bracket,

$$(3.4) \quad \xi'(\{a, b\}) = \{\xi(a), b\} + \{a, \xi(b)\}$$

for any $a, b \in A/J \subset A$. We claim that $\xi' = H_f$ for some $f \in J$. Indeed, the Poisson embedding $A/J \cong W_d \subset A$ induces a W_d -module structure on $J \subset A$ and a flat connection ∇ on the W_d -module J . Since the Poisson bivector Θ on $A/J \cong W_d$ is non-degenerate, we have

$$\xi' = \alpha \lrcorner \Theta$$

for some 1-form $\alpha \in J \otimes_{W_d} \Omega^1(W_d/k)$. Then the equality (3.4) means exactly that the form α is closed with respect to the connection ∇ , $\nabla \alpha = 0$. By Lemma 3.2, all the higher de Rham cohomology groups of the flat module J are trivial, so that $\nabla \alpha = 0$ implies that $\alpha = \nabla f$ for some $f \in J$. Thus in turn means that we indeed have

$$\xi' = df \lrcorner \Theta = H_f.$$

Replacing $\xi : A \rightarrow A$ with $\xi - H_f : A \rightarrow A$, we obtain a derivation that still satisfies (1.3), but now also preserves $A/J \subset A$. Therefore it also preserves $B \subset A$, and induces a derivation on B satisfying (1.3). This means that B is exact. \square

Remark 3.4. This Proposition is well-known in the theory of Poisson structures on C^∞ -manifolds, see [We]; the decomposition in this case is called the *Weinstein decomposition*. Our proof is essentially the same as Weinstein's, but it is re-set in the algebraic language and works for singular varieties, too.

Proof of Theorem 2.3. Almost all the claims follow immediately from Proposition 3.1 and Proposition 3.3. In Proposition 3.3 we take $A = \widehat{\mathcal{O}}_{X,x}$, the algebra of formal germs on functions on X near $x \in X$. The transversal slice \mathcal{Y}_x is the spectrum of the algebra B provided by Proposition 3.3. To prove that it is a symplectic variety, let \widehat{Y} be a resolution of \mathcal{Y}_x , and consider the product $\widetilde{Y} \times \widehat{X}_{i,x}^o$ as a resolution of \widehat{X}_x . Then, since the product decomposition (2.1) is Poisson, the symplectic form Ω on this product satisfies

$$\Omega = p_1^* \Omega_Y + p_2^* \Omega_{X_i},$$

where $p_1 : \widetilde{Y} \times \widehat{X}_{i,x}^o \rightarrow \mathcal{Y}_x$, $p_2 : \widetilde{Y} \times \widehat{X}_{i,x}^o \rightarrow \widehat{X}_{i,x}^o$ are the natural projections, and Ω_Y , Ω_{X_i} are the symplectic forms on \mathcal{Y}_x and $\widehat{X}_{i,x}^o$. Since the forms Ω and Ω_{X_i} have no poles, the form Ω_Y has no poles either. \square

4 Group actions.

We now turn to the proof of Theorem 2.4. By Theorem 2.3 and Theorem 2.5, we may assume that we are in the following situation:

- We have $\mathcal{Y}_x = \text{Spec } A$, where A is a complete local Poisson algebra, whose maximal ideal $\mathfrak{m} \subset A$ is preserved by all derivations of the algebra A . Moreover, the Poisson algebra A is exact.

We will say that a derivation ξ of a Poisson algebra B is *dilating with constant θ* if it satisfies

$$(4.1) \quad \xi(\{a, b\}) = \{\xi(a), b\} + \{a, \xi(b)\} + \theta\{a, b\}$$

for every $a, b \in B$. Since our algebra A is exact, there exist derivations $\xi : A \rightarrow A$ dilating with constant 1. Fix such a derivation. As explained in the proof of Lemma 2.2, to prove Theorem 2.4 we essentially need to find a derivation which is (1) dilating with non-zero constant and (2) can be integrated to an action of \mathbb{G}_m on the algebra A . This we will restate and prove in the following equivalent algebraic form.

Proposition 4.1. *In the assumptions of (•) above, there exists an integer $l \neq 0$ and a multiplicative grading*

$$A = \bigoplus_p A^p$$

on the algebra A such that

$$\{A^p, A^q\} \subset A^{p+q-l}.$$

Remark 4.2. We may assume that l is positive with any loss of generality. It would be highly desirable to show that one can choose a grading in Proposition 4.1 which only has positive weights, $A^p = 0$ for $p < 0$. Unfortunately, we were unable to prove it – it seems that this would require a radically different approach (perhaps a study of generalized contact singularities would help, see Remark 2.6). Thus, Proposition 4.1 is of only limited use in geometric application.

Proof. By (•), all the derivations of the algebra A preserve the maximal ideal $\mathfrak{m} \subset A$ and all its powers $\mathfrak{m}^q \subset A$. In particular, all the ideals $\mathfrak{m}^p \subset A$

are Poisson. Thus for every integer $q \geq 1$ the Artin algebra $A_q = A/\mathfrak{m}^q$ is a Poisson algebra. We have two canonical elements

$$m, p \in A_q \otimes_k A_q \otimes_k A_q^*$$

– namely, m defines the multiplication in A_q , and p defines the Poisson bracket.

Lemma 4.3. *Let B be a finite-dimensional Poisson algebra over k with*

$$m, p \in B \otimes_k B \otimes_k B^*$$

giving the multiplication and the Poisson bracket. An endomorphism $\xi \in \text{End}(B)$ of the vector space B is a dilating derivation of the algebra B with constant θ if and only if

$$\xi(m) = 0 \quad \xi(p) = \theta p,$$

where ξ acts on $B \otimes B \otimes B^$ via the canonical representation of the Lie algebra $\text{End}_k(B)$.*

Proof. Clear. □

Lemma 4.4. *Let $\xi \in \text{End}(B)$ be a dilating derivation of a finite-dimensional Poisson algebra B with constant θ . Let $\xi = \xi_s + \xi_n$ be its Jordan decomposition into the semisimple part ξ_s and the nilpotent part ξ_n .*

Then both ξ_s and ξ_n are also derivations. Moreover, ξ_s is dilating with constant θ , and ξ_n is dilating with constant 0 (in other words, preserves the Poisson structure). In particular, $\xi_s \neq 0$.

Proof. By the standard Lie algebra theory, the Jordan decomposition $\xi = \xi_s + \xi_n$ is universal – namely, it induces the Jordan decomposition of the endomorphism $\text{ad}(\xi) : V \rightarrow V$ for any finite-dimensional representation V of the reductive Lie algebra $\text{End}_k(B)$. In particular this applies to the representation $B \otimes B \otimes B^*$. Now, by assumption m and p are both eigenvectors of the endomorphism $\text{ad}(\xi)$, with eigenvalues respectively 0 and θ . Therefore $\text{ad}(\xi_n)(m) = \text{ad}(\xi_n)(p) = 0$, and both m and p are eigenvectors of $\text{ad}(\xi_s)$ with eigenvalues respectively 0 and θ . □

Now, by (\bullet) the fixed dilating derivation $\xi : A \rightarrow A$ preserves the ideal $\mathfrak{m}^q \subset A$ for any $q \geq 1$, so that we have a dilating derivation ξ of every quotient $A_q = A/\mathfrak{m}^q$.

Denote by $T_q \subset GL(A_q)$ the minimal algebraic subgroup whose Lie algebra $Lie(T_q) \subset \text{End}(A_q)$ contains the semisimple part ξ_s of the dilating derivation $\xi : A_q \rightarrow A_q$. Since the endomorphism ξ_s is semisimple and non-trivial, the group T_q is a non-trivial torus. All eigenvectors of the derivation ξ_s in any representation of group $GL(A_q)$ are also eigenvectors of the torus T_q ; in particular, this applies to the multiplication element $m \in A_q \otimes A_q \otimes A_q^*$ and to the Poisson bracket element $p \in A_q \otimes A_q \otimes A_q^*$. Since ξ_s is dilating with non-trivial constant, the torus T_q acts on the line $k \cdot p \subset A_q \otimes A_q \otimes A_q^*$ by a non-trivial character $\chi : T \rightarrow \mathbb{G}_m$.

Lemma 4.5. *For every $q > r \geq 1$, the torus T_q preserves the subspace $\mathfrak{m}^r \subset A_q$, and the corresponding reduction map $\text{red} : T_q \rightarrow GL(A_r)$ induces a group isomorphism $T_q \cong T_r$.*

Proof. Let $\xi_s, \xi_n \in \text{End}(A_q)$ be the semisimple and the nilpotent part of the endomorphism $\xi : A_q \rightarrow A_q$. By (\bullet) the vector field ξ preserves the maximal ideal $\mathfrak{m} \subset A_q$. Since $\text{codim}_k \mathfrak{m} = 1$, this is equivalent to preserving the corresponding line in the dual space A_q^* . Therefore by universality of the Jordan decomposition the endomorphisms ξ_s and ξ_n also preserve $\mathfrak{m} \subset A_q$. Since by Lemma 4.4 both ξ_s and ξ_n are derivations, they also preserve the ideal $\mathfrak{m}^r \subset A_q$ and act naturally on the quotient $A_r = A_q/\mathfrak{m}^r$.

Denote their reductions mod \mathfrak{m}^r by $\bar{\xi}_s, \bar{\xi}_n \in \text{End}(A_r)$. Then $\bar{\xi}_s$ is obviously semisimple, $\bar{\xi}_n$ is nilpotent, they commute, and $\bar{\xi}_s + \bar{\xi}_n$ is the given derivation $\xi : A_r \rightarrow A_r$. By the unicity of the Jordan decomposition, this means that $\bar{\xi}_s$ is actually the semisimple part of the endomorphism $\xi : A_r \rightarrow A_r$.

Denote by $P_{qr} \subset GL(A_q)$ the subgroup of endomorphisms which preserve the ideal $\mathfrak{m}^r \subset A_q$, so that we have a natural reduction map $\text{red} : P_{qr} \rightarrow GL(A_r)$, and let $T_{qr} = \text{red}^{-1}(T_r) \subset P_{qr}$ be the preimage of the torus $T_r \subset GL(A_r)$ under the natural map $P_{qr} \rightarrow GL(A_r)$. Since $\xi_s : A_q \rightarrow A_q$ reduces to $\bar{\xi}_s : A_r \rightarrow A_r$, the Lie algebra of the subgroup $T_{qr} \subset GL(A_q)$ contains $\xi_s \in \text{End}(A_q)$. By definition of the group T_q this means $T_q \subset T_{qr} \subset P_{qr}$. Therefore T_q indeed preserves $\mathfrak{m}^r \subset A_q$, and the natural reduction map $\text{red} : T_q \rightarrow GL(A_r)$ maps T_q into the subgroup $T_r \subset GL(A_r)$.

But the Lie algebra of the image $\text{red}(T_q) \subset T_r$ contains the endomorphism $\bar{\xi}_s : A_r \rightarrow A_r$. Therefore by definition $\text{red}(T_q) = T_r$, in other words, the map $\text{red} : T_q \rightarrow T_r$ is surjective.

To prove that this map is injective, it suffices to prove that the corresponding Lie algebra map is injective. Let $a \in \text{End}(A_q)$ be an element in the Lie algebra of the torus T_q such that $\text{red}(a) = 0$. Since T_q is a torus,

a must be semisimple. On the other hand, a must be a derivation of the algebra A_q which is zero mod \mathfrak{m}^r – in other words, it must send the whole A_q into $\mathfrak{m}^r \subset A_q$. This implies that the endomorphism $a : A_q \rightarrow A_q$ is nilpotent. We conclude that $a = 0$. \square

To finish the proof of Proposition 4.1, it suffices to pass to the inverse limit. We see that there exists a non-trivial torus $T = T_q$, $q \geq 1$ which acts on

$$A = \varprojlim A_q,$$

and a non-trivial character $\chi : T \rightarrow \mathbb{G}_m = k^*$, such that

$$t(ab) = t(a)t(b) \quad t(\{a, b\}) = \chi(t)\{t(a), t(b)\}$$

for every $t \in T$, $a, b \in A_\infty$. Take an embedding $\tau : \mathbb{G}_m \rightarrow T$ such that $\chi \circ \tau(a) = a^l$ for some non-trivial integer l , and define the grading by means of the induced \mathbb{G}_m -action on A . \square

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STEKLOV MATH INSTITUTE
MOSCOW, USSR

E-mail address: kaledin@mccme.ru